CH.8/TORSION

Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear Figure (1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory, to make the following basic assumptions:

(1) The material is homogeneous, i.e. of uniform elastic properties throughout.

(2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.

(3) The stress does not exceed the elastic limit or limit of proportionality.

(4) Circular Sections remain circular.

(5) Cross-sections remain plane. (This is certainly not the case with the torsion of non circular Sections.)

(6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.

Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

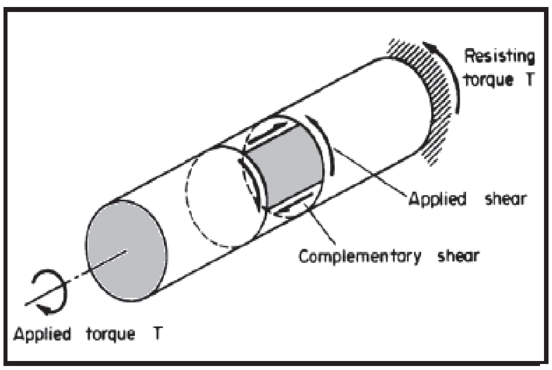

Figure (1) Shear system set up on an element in the surface of a shaft subjected to torsion.

(a) Angle of twist

Consider now the solid circular shaft of radius (R) subjected to a torque (T) at one end, the other end being fixed Figure (2). Under the action of this torque a radial line at the free end of the shaft twists through an angle (θ), point A moves to B, and AB subtends an angle (γ) at the fixed end. This is then the angle of distortion of the shaft, i.e. the shear strain.

angle in radians = arc /radius

$$\text{arc } AB = R\theta = L\gamma$$

$$\therefore \quad \gamma = R\theta/L \qquad \text{... (1)}$$

From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

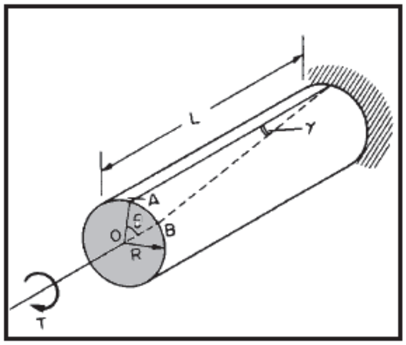

Figure (2)

$$\therefore \quad \gamma = \frac{\tau}{G} \qquad \text{....(2)}$$

where τ is the shear stress set up at radius R.

Therefore equating eqns. (1) and (2),

$$\frac{R\theta}{L} = \frac{\tau}{G}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right) \qquad \text{....(3)}$$

where τ' is the shear stress at any other radius r.

(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness (dr) as shown in Figure (3) each subjected to a shear stress (τ').

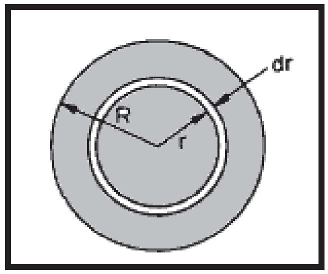

Figure (3) Shaft cross-section

The force set up on each element,

= stress x area= τ' x $2.\pi.r$ dr (approximately)

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

= (τ' x $2.\pi.r$ dr).r= τ' x $2.\pi.r^2$ dr

The total torque on the section (**T**) will then be the sum of all such contributions across the section,

$$T = \int_0^R 2\pi\tau' r^2 \, dr$$

Now the shear stress (**τ'**) will vary with the radius rand must therefore be replaced in terms of ***r*** before the integral is evaluated. From eqnuation (3)

$$\tau' = \frac{G\theta}{L} r$$

$$T = \int_0^R 2\pi \frac{G\theta}{L} r^3 \, dr$$

$$= \frac{G\theta}{L} \int_0^R 2\pi r^3 \, dr$$

The integral $\int_0^R 2.\pi.r^3.dr$ is called the polar second moment of area (J), and may be evaluated as a

standard form for solid and hollow shafts .

$$\therefore \qquad T = \frac{G\theta}{L} J$$

$$\text{or} \qquad \frac{T}{J} = \frac{G\theta}{L} \qquad \text{....(4)}$$

Combining eqns. ***(3)*** and ***(4)*** produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \qquad \text{.....(5)}$$

Polar second moment of area

As stated above the polar second moment of area ***J*** is defined as

$$J = \int_0^R 2\pi r^3\, dr$$

For a *solid shaft,*

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32} \qquad \text{.... (6)}$$

For a hollow ***shaft*** of internal radius ***r,***

$$J = 2\pi \int_r^R r^3\, dr = 2\pi \left[\frac{r^4}{4}\right]_r^R$$

$$= \frac{\pi}{2}(R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32}(D^4 - d^4) \qquad \text{.... (7)}$$

For thin-walled hollow shafts the values of (D) and (d) may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

Now

$$J = \int_0^R 2\pi r^3\, dr = \Sigma(2\pi r\, dr)r^2$$
$$= \Sigma\, Ar^2$$

Where; A = 2πr dr is the area of each small element of Figure (3).
If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness t = dr, then

$$J = Ar^2 = (2\pi r\, t)r^2$$
$$= 2\pi r^3 t \text{ (approximately)} \qquad .... (8)$$

Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Figure (1),their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L}R$$

Now from the definition of the shear or rigidity modulus (**G),**

$$\tau = G\gamma$$

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L}R = G\gamma \qquad .... (9)$$

or, in terms of some internal radius ***r*,**

$$\tau' = \frac{G\theta}{L}r = G\gamma \qquad .... (10)$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius Figure (4) .

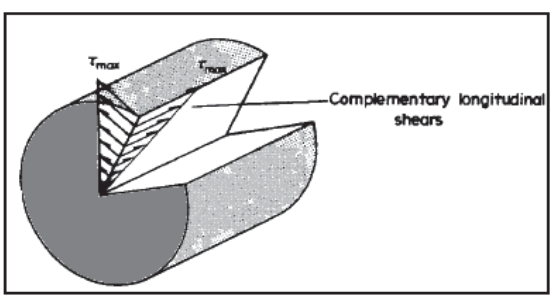

Figure (4) Complementary longitudinal shear stress in a shaft subjected to torsion.

Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{TR}{J}$$

With (R) the outside radius of the shaft the above equation yields the greatest value possible for **T,** Figure (4).

$$\tau_{max} = \frac{TR}{J}$$

$$\tau_{max} = \frac{T}{Z} \qquad \text{.... (11)}$$

Where; z = J/R is termed the polar section modulus. It will be seen from the preceding section that:

for solid shafts, $$Z = \frac{\pi D^3}{16} \qquad \text{..... (12)}$$

and for hollow shafts, $$Z = \frac{\pi(D^4 - d^4)}{16D} \qquad \text{.... (13)}$$

Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L} = \frac{T}{GJ}$$

The quantity (GJ) is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ = \frac{T}{\theta/L} \qquad \text{.... (14)}$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

Composite shafts - series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed ***series-connected*** Figure (5) .

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \qquad .... (15)$$

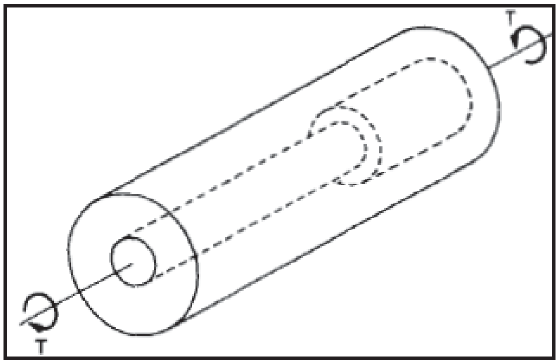

Figure (5)"Series-connected" shaft – common torque.

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \qquad .... (16)$$

Composite shafts - parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel Figure (6).

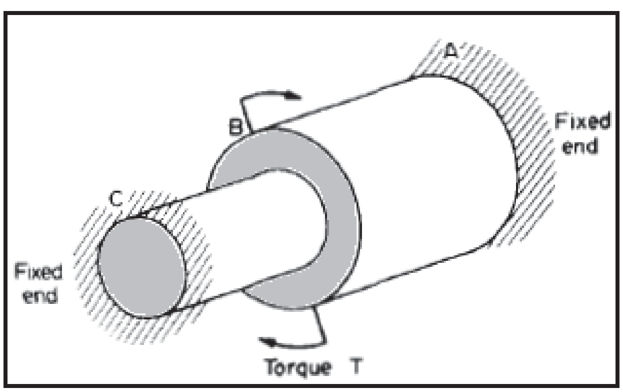

Figure (6)"Parallel-connected" shaft – shared torque.

For parallel connection,

total torque $T = T_1 + T_2$ ....(17)

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad \text{.....(18)}$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \quad \text{....(19)}$$

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

Strain energy in torsion

The strain energy stored in a solid circular bar or shaft subjected to a torque (T) is given by the alternative expressions.

$$U = \tfrac{1}{2} T\theta = \frac{T^2 L}{2GJ} = \frac{GJ\theta^2}{2L} = \frac{\tau^2}{4G} \times \text{volume} \quad \text{.....(20)}$$

Power transmitted by shafts

If a shaft carries a torque T Newton metres and rotates at ***o*** rad/s it will do work at the rate of;

T.ω Nm/s (or joule/s).

Now the rate at which a system works is defined as its power, the basic unit of power being the

Watt (1 Watt = 1 N.m/s).

Thus, the power transmitted by the shaft:

= T.ω Watts.

Since the Watt is a very small unit of power in engineering terms use is normally made of SI. multiples, i.e. kilowatts (kW) or megawatts (MW).

Combined bending and torsion - equivalent bending moment

For shafts subjected to the simultaneous application of a bending moment (**M*)*** and torque (T) the principal stresses set up in the shaft can be shown to be equal to those produced by an ***equivalent bending moment,*** of a certain value (**M_e**) acting alone.

From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment (M) are given by

$$\sigma = \frac{My_{max}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft J = 2I,

$$\tau = \frac{TD}{4I}$$

The principal stresses for this system can now be obtained by applying the formula derived in

$$\sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

and, with $\sigma_y = 0$, the maximum principal stress (σ_1) is given by

$$\sigma_1 = \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left[\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2\right]}$$

$$= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

Now if (M_e) is the bending moment which, acting alone, will produce the same maximum stress, then

$$\sigma_1 = \frac{M_e y_{max}}{I} = \frac{M_e D}{2I}$$

$$\frac{M_e D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

i.e. the equivalent bending moment is given by

$$M_e = \tfrac{1}{2}[M + \sqrt{(M^2 + T^2)}] \qquad \text{.....(21)}$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.

Combined bending and torsion - equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment (M) and a torque (T) the ***maximum shear stress*** set up in the shaft may be determined by the application of an ***equivalent torque*** of value (T_e) acting alone. From the preceding section the principal stresses in the shaft are given by

$$\sigma_1 = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M + \sqrt{(M^2 + T^2)}]$$

$$\sigma_2 = \frac{1}{2}\left(\frac{D}{2I}\right)[M - \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M - \sqrt{(M^2 + T^2)}]$$

Now the maximum shear stress is given by equation (12)

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\left(\frac{D}{J}\right)\sqrt{(M^2 + T^2)}$$

But, from the torsion theory, the equivalent torque T_e , will set up a maximum shear stress of

$$\tau_{max} = \frac{T_e D}{2J}$$

Thus if these maximum shear stresses are to be equal,

$$T_e = \sqrt{(M^2 + T^2)} \qquad .....(22)$$

Examples

Example 1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if G = 80 GN/m^2.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution

(a) Power $= T\omega$ $\therefore$ torque $T = \dfrac{\text{power}}{\omega}$

$$\therefore \quad T = \frac{75 \times 10^3}{150 \times 2\pi/60} = 4.77 \text{ kN m}$$

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{and} \quad J = \frac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6} \text{ m}^4$$

$$\therefore \tau_{max} = \frac{TR_{max}}{J} = \frac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = \mathbf{24.3 \ MN/m^2}$$

Also from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m}$$

$$= 6.07 \times 10^{-3} \times \frac{360}{2\pi} = \mathbf{0.348 \ degrees/m}$$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}(100^4 - 60^4)10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$$

The torque carried by the modified shaft is then given by

$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = \mathbf{4.15 \times 10^3\ Nm}$$

Now, weight/metre of original shaft

$$= \frac{\pi}{4}(100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3}\ \rho g$$

where ρ is the density of the shaft material.

$$\text{Also, weight/metre of modified shaft} = \frac{\pi}{4}(100^2 - 60^2)10^{-6} \times 1 \times \rho g$$

$$= 5.027 \times 10^{-3}\ \rho g$$

$$\text{Power/weight ratio for original shaft} = \frac{T\omega}{\text{weight/metre}}$$

$$= \frac{4.77 \times 10^3\ \omega}{7.854 \times 10^{-3}\ \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3\ \omega}{5.027 \times 10^{-3}\ \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = \mathbf{36\%}$$

Example 2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8^o in a length of 4 m. For the shaft material G = 80 GN/m^2.

Solution

Maximum shear stress condition

Since power $= T\omega$ and $\omega = 200 \times \frac{2\pi}{60} = 20.94$ rad/s

$$T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3\ \text{Nm}$$

From the torsion theory $J = \frac{TR}{\tau}$

$$\therefore \quad \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

But $d/D = 0.75$

$$\therefore \frac{\pi}{32} D^4 (1 - 0.75^4) = 20.43 \times 10^{-6} D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

$$d = 50.5 \text{ mm}$$

Angle of twist condition

Again from the torsion theory $J = \dfrac{TL}{G\theta}$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4 (1 - 0.75^4) = 2.156 \times 10^{-6}$$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4 (1 - 0.75^4) = 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

$$d = 56.5 \text{ mm}$$

Thus the dimensions required for the shaft to satisfy both conditions are outer diameter 75.3mm; inner diameter 56.5 mm.

Example 3

(a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m^2.

(b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

Solution

(a) This is, in effect, a question on shafts in series since each part is subjected to the same torque. From the torsion theory ;

$$T = \frac{\tau J}{R}$$

and as the maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts the torque allowable for a known value of shear stress is dependent only on the value of (J). This will be least where the internal diameter is greatest since

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\text{least value of } J = \frac{\pi}{32}(50^4 - 38^4)10^{-12} = 0.41 \times 10^{-6}\,\text{m}^4$$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m² (a 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3\,\text{Nm}$$

$$\text{Maximum power} = T\omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60}$$

$$= 25.2 \times 10^3 = \mathbf{25.2\,kW}$$

(b)

Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, equation (16) applies,

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32}(50^4 - 25^4)10^{-12}}{\frac{\pi}{32}(50^4 - 38^4)10^{-12}} = 1.43$$

$$L_2 = 1.43\,L_1$$

$$L_1 + L_2 = 510\,\text{mm}$$

$$L_1(1 + 1.43) = 510$$

$$L_1 = \frac{510}{2.43} = \mathbf{210\,mm}$$

PROBLEMS

1 - A solid steel shaft (A) of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m^2 in the steel.
It is proposed to replace (A) by a hollow shaft (B), of the Same external diameter but with a limiting shearing stress of 75 MN/m^2. Determine the internal diameter of (B) to transmit the same power at the same speed. **Ans.[38.6kW, 33.4 mm]**

2 - Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20 % and the greatest intensity of shear stress is limited to75 MN/m^2. The internal diameter of the shaft is to be 80 % of the external diameter. (The mean torque is that derived from the horsepower equation.)
Ans. [135.2mm, 108.2 mm.]

3 - A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the
shaft are:
(a) that the shaft must not twist more than 0.02 radian on a length of 10 diameters;
(b) that the working stress must not exceed 60 MN/m^2.
If the modulus of rigidity of steel is 80 GN/m^2 what is
(i) the diameter of the shaft required
(ii) the actual working stress;
(iii) the angle of twist of the 3 m length? **Ans. [l50 mm; 60MN/m^2; 0.030 rad.]**

4 - A hollow shaft has to transmit 6MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m^2 and the angle of twist 0.3^o per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. G = 80 GN/m^2. **Ans. [61.5mm.]**

5 - A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m^2 and 55 MN/m^2 respectively, and the maximum torque is 1.4 times the mean torque, calculate:
(a) the diameter of the shaft;
(b) the diameter of the bolts. **Ans. [67.2mm, 13.8 mm.]**

6 - A hollow low carbon steel shaft is subjected to a torque of 0.25 MN. m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m^2 determine the required diameters and the angle of twist in degrees per metre length of shaft.
G = 80GN/m^2. **Ans. [264mm, 88 mm; 0.38^o]**